RECITATION 7

IMPLICIT DIFFERENTIATION

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Section 1. *Exercises*

Exercise 1

Suppose $y^2 + 2xy + x^2 = 5$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y and x.

Solution .:.

Implicit differentiation yields that $2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y + 2x = 0$. Hence solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x-y}{x+y} = -1$ for $x + y \neq 0$.

Exercise 2

Differentiate $f(x) = x^{2x}$ using implicit differentiation.

Solution .:.

Taking the natural log of both sides, we have that $\ln(f(x)) = 2x\ln(x)$. Implicit differentiation the yields $\frac{1}{f(x)}f'(x) = 2\ln(x) + 2x\frac{1}{x}$, equivalently,

$$f'(x) = (2\ln(x) + 2)f(x) = (2\ln(x) + 2)x^{2x}.$$

Exercise 3

Find the tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$ using implicit differentiation.

Solution .:.

Implicit differentiation yields that $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{2y} = -x/y$ for $y \neq 0$. At $(3, 4)$, the tangent line then has slope $-3/4$. Given the point $(3, 4)$, this yields the line described by

$$y - 4 = \frac{-3}{4}(x - 3).$$

Exercise 4

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y and x, where $\sin(y) = x$.

Solution .:.

Implicit differentiation yields that $\cos(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 1$ and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos(y)} = \sec(y)$.

Exercise 5

Consider $x \cdot y = x + y$. Where is the tangent line horizontal?

Solution .:.

The tangent line is horizontal iff $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Using implicit differentiation, we have $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$, and thus $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{1-x}$. This is 0 iff $y = 1$, i.e. $x = x + 1$, which is impossible. Therefore the tangent line is never horizontal.

Exercise 6

Take $y^2 + 2y + 4 = x^2 + 2x + 4$. Find $\frac{dy}{dx}$ in terms of x and y.

Solution .:.

Using implicit differentiation, $\frac{dy}{dx} \cdot (2y + 2) = 2x + 2$. Hence $\frac{dy}{dx} = \frac{2x+2}{2y+2} = \frac{x+1}{y+1}$.

Exercise 7

Take $e^y = x$. We know that then $y = \ln(x)$ has $\frac{dy}{dx} = 1/x$. Prove this using implicit differentiation.

Solution .:.

Using implicit differentiation, $e^y \frac{dy}{dx} = 1$, meaning $\frac{dy}{dx} = 1/e^y = 1/x$.

Section 2. The Midterm

Question 1

Evaluate the limit or determine it does not exist: $\lim_{x \to 3} \left(\dfrac{x^2 - 2x - 3}{x - 3} \right)$.

Solution .:.

This limit is equal to

$$\lim_{x \to 3} \left(\frac{x^2 - 2x - 3}{x - 3} \right) = \lim_{x \to 3} \frac{(x-3)(x+1)}{x-3} = \lim_{x \to 3} (x+1) = 3 + 1 = 4.$$

Question 2

Find the x-coordinate of each point on the graph of $y = (x^2 - 15)e^x$ where the tangent line is horizontal. Show all work.

Solution .:.

The slope of the tangent line is just $\frac{dy}{dx}$. So the tangent line is horizontal iff $\frac{dy}{dx} = 0$. The product rule yields that $\frac{dy}{dx} = 2xe^x + (x^2 - 15)e^x = (x^2 + 2x - 15)e^x$. This is 0 iff $x^2 + 2x - 15$ is 0, because e^x is never 0. Note that $x^2 + 2x - 15 = (x+5)(x-3)$ so that this is 0 iff $x = -5$ or $x = 3$.

Question 3

Calculate $f'(x)$ after calculating the derivative, do not simplify your answer: $f(x) = \ln(x)\tan(x)$

Solution .:.

By the product rule, $f'(x) = \ln(x) \cdot \frac{d}{dx}\tan(x) + \tan(x) \cdot \frac{d}{dx}\ln(x)$. We know $\frac{d}{dx}\ln(x) = \frac{1}{x}$, and $\frac{d}{dx}\tan(x) = \sec^2(x)$ (which can be derived by the quotient rule along with the derivatives of sine and cosine). Hence $f'(x) = \ln(x)\sec^2(x) + \tan(x)/x$.

Question 4

Evaluate the limit or determine it does not exist: $\lim_{x \to 2} \left(\dfrac{\sqrt{x+7} - 3}{x - 2} \right)$.

Solution .:.

Multiplying and dividing by the conjugate, we get

$$\begin{aligned}\lim_{x\to 2}\left(\frac{\sqrt{x+7}-3}{x-2}\right) &= \lim_{x\to 2}\left(\frac{|x+7|-9}{(x-2)(\sqrt{x+7}+3)}\right)\\ &= \lim_{x\to 2}\left(\frac{x-2}{(x-2)(\sqrt{x+7}+3)}\right)\\ &= \lim_{x\to 2}\left(\frac{1}{\sqrt{x+7}+3}\right)\\ &= \frac{1}{\sqrt{9}+3} = \frac{1}{6}.\end{aligned}$$

Question 5

Evaluate the limit or determine it does not exist: $\lim_{x\to 0}\left(\frac{3x}{\sin(7x)}\right)$.

Solution .:.

Since $\lim_{x\to 0}\sin(x)/x = 1$ and $\lim_{x\to 0} 7x = 0$, it follows that $\lim_{x\to 0}\sin(7x)/(7x) = 1$. Therefore,

$$\begin{aligned}\lim_{x\to 0}\left(\frac{3x}{\sin(7x)}\right) &= \lim_{x\to 0}\left(\frac{7x}{\sin(7x)}\cdot\frac{3}{7}\right)\\ &= \frac{3}{7}\lim_{x\to 0}\left(\frac{7x}{\sin(7x)}\right) = \frac{3}{7}.\end{aligned}$$

Question 6

Calculate $f'(x)$. After calculating the derivative, do not simplify your answer: $f(x) = \sqrt{\cos(10+x^4)}$.

Solution .:.

$f'(x) = \frac{1}{2}\cos(10+x^4)\cdot(-\sin(10+x^4))\cdot(4x^3)$.

Question 7

Calculate $f'(x)$. You must use the limit definition of derivative to recieve any credit. Show all work: $f(x) = 2x - x^3$.

Solution .:.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{2(x+h)-(x+h)^3-(2x-x^3)}{h}\\ &= \lim_{h\to 0}\frac{2x+2h-(x^3+3x^2h+3xh^2+h^3)-2x+x^3}{h}\\ &= \lim_{h\to 0}\frac{2h-3x^2h-3xh^2-h^3}{h}\\ &= \lim_{h\to 0} 2-3x^2-3xh-h^2\\ &= 2-3x^2-0-0 = 2-3x^2.\end{aligned}$$

Question 8

Find all solutions, if any, to the following equation:

$$\ln\left(\frac{x^2}{3-x}\right) - \ln(x) = \ln\left(\frac{4x}{10+x}\right).$$

Solution .:.

Note that $\ln\left(\frac{x^2}{3-x}\right) - \ln(x) = \ln\left(\frac{x}{3-x}\right)$ for all x where these are defined. Therefore, if the equation holds,

$$\ln\left(\frac{x}{3-x}\right) = \ln\left(\frac{4x}{10+x}\right)$$
$$\to \frac{x}{3-x} = \frac{4x}{10+x}.$$

Since the equation holds, $\ln(x)$ is defined, i.e. $x \neq 0$. Therefore, this implies

$$\to \frac{1}{3-x} = \frac{4}{10+x}.$$
$$\to 10 + x = 12 - 4x \to 5x = 2 \to x = 2/5.$$

And we can check that this holds rather easily.

Question 9

Find the absolute minimum value and absolute maximum value of f on the given interval. Show all work: $f(x) = (4-x)e^x$ on $[0, 4]$. Hint: $2 < e < 3$.

Solution .:.

By the product rule, $f'(x) = -e^2 + (4-x)e^x = (3-x)e^x$. This is 0 iff $3 - x = 0$, i.e. $x = 3$. Since $f'(x)$ is always defined, $x = 3$ is the only critical point of f. So we merely need to test the endpoints and $x = 3$:

$$f(0) = (4-0)e^0 = 4 \qquad f(3) = (4-3)e^3 = e^3 \qquad f(4) = (4-4)e^4 = 0.$$

So clearly 0 is the minimum value of f on $[0, 4]$ (and it occurs at $x = 4$). Since $e^3 > 2^3 > 4$, e^3 is the maximum value of f on $[0, 4]$ (and it occurs at $x = 3$).

Question 10

Find all the critical numbers of $f(x) = x - \ln(x)$ or determine that there are no critical numbers [the professor write "critical number" instead of the more standard "critical point"].

Solution .:.

$f'(x) = 1 - \frac{1}{x}$. This is 0 iff $1 = 1/x$, i.e. $x = 1$. $f'(x)$ is undefined only when $x = 0$, but this isn't in the domain of f' anyway, so it's not a critical point. Hence $x = 1$ is the only critical point.

Question 11

Find the value of k that makes f continuous at $x = -2$ or determine that no such value of k exists. Show all work. You must use limits and proper notation to receive full credit:

$$f(x) = \begin{cases} 3x + k & \text{if } x < -2 \\ 4 & \text{if } x = -2 \\ kx^3 + 3 & \text{if } x > -2. \end{cases}$$

Solution .:.

f will be continuous at -2 iff $f(-2) = \lim_{x\to-2} f(x)$, which then requires $\lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} f(x) = f(-2)$. Note that

$$f(-2) = 4$$

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} 3x + k = k - 6$$
$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} kx^3 + 3 = -8k + 3.$$

Hence we require $k - 6 = 4 = -8k + 3$. But $k - 6 = 4$ implies $k = 10$ while $-8k + 3 = 4$ implies $k = 1/8 \neq 10$. Therefore, no such value of k can exist, and f can never be continuous at -2.

Question 12

Find an equation of the line tangent to the graph $y = x^3 - 4x^2 + 5x + 1$ at $x = 1$. Show all work. You may give any form of the equation of a line.

Proof .:.

When $x = 1$, $y = 1 - 4 + 5 + 1 = 3$. So we have the point $(1, 3)$. All that we require is the slope of the line, which is given by $f'(1)$:

$$f'(x) = 3x^2 - 8x + 5 \to f'(1) = 3 - 8 + 5 = 0.$$

Hence by point-slope form, the equation of the tangent line can be written $y - 3 = 0 \cdot (x - 1)$, or $y = 3$.

Question 13

Calculate $f'(x)$. After calculating the derivative, do not simplify your answer: $f(x) = \left(\frac{x^3-x}{x^2+7}\right)^{2/3}$.

Proof .:.

Using the chain and quotient rules,

$$f'(x) = \frac{2}{3}\left(\frac{x^3 - x}{x^2 + 7}\right)^{-1/3} \cdot \left(\frac{(3x^2 - 1)\cdot(x^2 + 7) - (x^3 - x)\cdot(2x)}{(x^2 + 7)^2}\right).$$